An Efficient Method for Approximation of Non-Rational Transfer Functions

Tomislav B. Šekara, Milan R. Rapaić, and Mihailo P. Lazarević

***Abstract*—A method for rational approximation of linear fractional order systems (LFOS) is presented in the present paper. The method is computationally efficient, flexible and effective, as is illustrated by numerous examples. The proposed approach can also be used as an intermediate stage in designing indirect discrete rational approximations.**

Index Terms—Discretization, fractional order systems, rational approximations.

Original Research Paper
DOI: 10.7251/ELS1317040S

I. Introduction

DESIGNING classical and/or fractional order control laws involving integral and differential actions [1, 2] often requires formulation of a discrete model of the process by using methods of invariable response to a pulse or Heaviside excitation and a series of other approximate methods [3-20]. Since a process can, in general, is represented by a transfer function $G_p(s)$ which is not a rational function [21, 22], the problem of rational approximation and discretization in general becomes complex. In addition, the fundamental system properties, such are steady-state gain and settling time, as well as basic properties in the frequency domain, must be preserved. In the process of discretization of LFOS, where, in general, fractional order integral and differential actions belong, one can make use of the well-known mapping of s-domain to z-domain in the complex plane

$$z = e^{sT} , \tag{1}$$

where T is the sampling time. Transform (1) maps left half-plane of the s-plane to interior of the unity circle in the z-plane. This means that stability of the discrete system has been preserved if all poles of the discrete system are located within the unity circle. One of the basic goals of discretization is acquiring the ability for practical realization of the corresponding control laws or of some other requirements in order that the digital model is fully equivalent to the continuous system over a wide frequency range.

The method proposed within the present paper relies on the interpolation of the frequency characteristic of the system on a predefined set of target frequencies. The approach has originally been proposed in [25]. The present manuscript extends the development of [25] by suggesting the possibility of using least-squares approximation on a wider frequency range.

For the purpose of illustration of practical importance of LFOS, consider a process described by classical diffusion equation (also referred to as the heat equation), which is ubiquitous in science and engineering since it simultaneously describes a number of transfer phenomena, including heat-transfer and a number of other diffusion-like processes. These diffusion-like processes include diffusion of mass (mechanical diffusion), diffusion of momentum (viscosity), diffusion of electrical potential (in long lines, when inductivity is negligible), and many others. One-dimensional diffusion equation is a partial differential equation of the form

$$\tau \frac{\partial^2 \rho}{\partial z^2} = \frac{\partial \rho}{\partial t} , \ \tau > 0 \tag{2}$$

describing the process of transport (diffusion) of a quantity ρ along the z axis in time t. For simplicity, let us address only the diffusion within a semi-infinite medium, where both space and time variable take arbitrary positive values. Let us assume also that the process can be controlled by acting on the cross-section $z = 0$, and that the process output is taken (measured) at the cross-section $z = L$. The dynamics of the process is influenced by the diffusion time constant $\tau = \tau(z, t)$, which is, in general, a function of both space and time. However, in a variety of practically interesting cases this coefficient can be approximated by a constant factor.

Without loss of generality, assume that (2) describes a heat conduction process schematically shown in Fig. 1. Let us obtain its transfer function. In this particular case, $\rho = \rho(z, t)$, is the temperature of the cross section defined by space coordinate z evaluated at time instant t. Let $\tilde{\rho} = \tilde{\rho}(z,s)$ denote the Laplace transform of ρ, where the Laplace transform is taken with respect to the time variable t and the space variable z is considered as a parameter,

Manuscript received 19 April 2013. Accepted for publication 8 May 2013.
The authors would like to gratefully acknowledge the financial support of Serbian Ministry of Science and Education, under grants TR33020 (T.B.Š), TR32018 and TR33013 (M.R.R) and III41006 (M.P.L.).
T. B. Šekara is with the Faculty of Electrical Engineering, University of Belgrade, Belgrade, Serbia (e-mail: tomi@etf.rs).
M. R. Rapaić is with Faculty of Technical Sciences, University of Novi Sad, Novi Sad, Serbia. (tel: +381621711982, e-mail: rapaja@uns.ac.rs).
M. P. Lazarević is with the Faculty of Mechanical Engineering, University of Belgrade, Belgrade, Serbia (e-mail: mlazarevic@mas.bg.ac.rs).

$$\tilde{\rho}(z,s) = \int_0^{\infty} \rho(z,t) e^{-st} dt . \quad (3)$$

By applying the Laplace transform to equation (2), one obtains general solution

$$\tilde{\rho}(s,z) = C_1(s)\, e^{-z\sqrt{s/\tau}} + C_2(s)\, e^{z\sqrt{s/\tau}} . \quad (4)$$

Since any heat conduction process is stable, the Laplace transform of the temperature in any cross-section must be bounded, i.e.

$$\lim_{z\to\infty} \tilde{\rho}(s,z) = const. \quad (5)$$

has to be satisfied, thus C_2=0 and equation (4) takes the form

$$\tilde{\rho}(s,z) = C_1(s)\, e^{-z\sqrt{s/\tau}} . \quad (6)$$

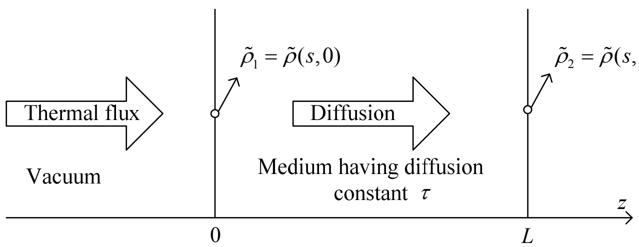

Fig. 1. A sketch of the process of heat conduction by diffusion.

Integration "constant" C_1 as well as the conduction function is determined from the known (or given) boundary conditions. In view of this, the most frequent cases in practice are:

Case 1. Heat conduction without any convective exchange of heat with the environment and fixed temperature at the "left" boundary. In this particular case, the temperature of the cross-section $z = 0$ could be controlled directly, and considered as the input of the process, while the dependent temperature of the cross-section $z = L$ could be considered as the output. The left boundary condition for this case is $\tilde{\rho}(s,0) = C_1(s)$, and the transfer function takes the form

$$G_a(s) = \frac{\tilde{\rho}_2(s,L)}{\tilde{\rho}_1(s,0)} = e^{-L\sqrt{s/\tau}} = e^{-\sqrt{Ts}} , \; T = L^2/\tau . \quad (7)$$

Case 2. Heat conduction without any convective exchange of heat with the environment and fixed thermal flux at the "left" boundary. The process is influenced by gradient of quantity ρ at z = 0 (this is the boundary surface of the medium of Fig. 1), the input quantity of the process being thermal flux through the boundary surface (again without any convective exchange with the environment)

$$\psi = -\lambda \left. \frac{d\tilde{\rho}(s,z)}{dz} \right|_{z=0} \quad (8)$$

and the process (output) quantity is $\rho_2 = \rho(s, \mathrm{L})$, and the transfer function is

$$G_b(s) = \frac{\tilde{\rho}_2}{\psi} = \frac{K}{\sqrt{s}} e^{-\sqrt{Ts}} , \; T = L^2/\tau , \; K = \sqrt{\tau}/\lambda . \quad (9)$$

Case 3. Heat conduction without any convective exchange of heat with the environment. The last characteristic case is when the convection is no longer neglected. Now, the process is influenced by a linear combination of the thermal flux and temperature at the "left" boundary

$$u = -\lambda \left. \frac{d\tilde{\rho}(s,z)}{dz} \right|_{z=0} + \eta\, \tilde{\rho}(s,z) , \quad (10)$$

with output $\rho_2 = \rho(s,L)$, and the transfer function is

$$G_c(s) = \frac{\tilde{\rho}_2}{u} = \frac{K}{1+\sqrt{T_1 s}} e^{-\sqrt{Ts}} , K = \frac{1}{\eta} , T_1 = \frac{\lambda^2}{\eta^2 \tau} , T = \frac{L^2}{\tau} . \quad (11)$$

In the examples above, the semi-derivative operator has appeared in a number of contexts. It should be mentioned that other forms of fractional order transfer functions emerge during investigations of different transfer phenomena. In the analysis of axial diffusion, i.e. diffusion from the axis of the cylinder towards its lateral surface or vice versa, one meets transfer functions originating from the Laplace transforms of Bessel functions, which have the form

$$G(s) = \frac{K}{\sqrt{1+sT}} \quad (12)$$

From this example, transfer functions given by equations (7), (9), (11), and (12) belong to the fractional order systems having transfer functions which belong to the class of irrational functions [23,24].

Since these transfer functions describe adequately physical processes, a logical question arises whether it is possible to formulate fractional order control laws and what would be their contribution to process control. Among many modern control strategies utilizing fractional order calculus, Podlubny's Fractional order PID [1,2] regulator is emphasized here. Classical PID is arguably the most utilized control strategy in use today. By replacing classical integral and differential actions by their respective fractional order analogues, the flexibility and applicability of the PID regulator can be greatly increased. Transfer function of the fractional order PID is of the form

$$\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}(s) = k + k_{\mathrm{i}} s^{-\lambda} + k_{\mathrm{d}} s^{\mu} , \; \lambda, \mu \in [0,1] . \quad (13)$$

The reader should notice that the implementation of Fractional order PID requires direct implementation of fractional order integrator and differentiator. Similar is also true for other

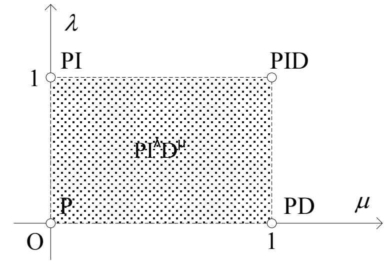

Fig. 2. A sketch of the process of heat conduction by diffusion.

types of fractional order regulators, as it can be seen from [25-27]. Such regulators are typically implemented as high order FIR or IIR filters [28], Realization of fractional order control laws involving an adequate discretization is possible thanks to the fast modern computers. It is known that in the regulator design two approaches are possible, direct design in the discrete domain and the other approach is design in the continuous domain first and then transition to the discrete domain. Obviously, discretization is required by both approaches. However, the discretization procedure is not straightforward when fractional order systems are in question, a problem which has been causing a considerable interest over the past years.

The paper is organized as follows. In the following Section II the proposed method for rational approximation of FOS has been outlined. Numerical examples are presented in Section III. Concluding remarks are presented in Section IV.

II. Rational Approximations of Transfer Functions of LFOS

Let us consider rational transfer function

$$\frac{B(s)}{A(s)}=\frac{b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0} \quad (14)$$

which should be used to approximate transfer function $G(s)$ of a linear fractional order system. For ($G(0)\neq 0$, b_0=1) or ($G(0)=0$, a_0=1) there are 2n real coefficients which should be determined from 2n equations obtained from the condition of overlapping the frequency characteristics in the corresponding discrete frequency points $\omega\in[\omega_0,\omega_2,\cdots,\omega_{n-1}]$, i.e.

$$G(\mathrm{i}\omega_k)-B(\mathrm{i}\omega_k)/A(\mathrm{i}\omega_k)=0,\ k=\overline{0,n-1},\ \mathrm{i}=\sqrt{-1}, \quad (15)$$

or for $G(0)\neq 0$, $b_0=1$ one obtains

$$\mathrm{Re}\left(G(\mathrm{i}\omega_k)A(\mathrm{i}\omega_k)-B(\mathrm{i}\omega_k)\right)=0,\ k=\overline{0,n-1}, \quad (16)$$

$$\mathrm{Im}\left(G(\mathrm{i}\omega_k)A(\mathrm{i}\omega_k)-B(\mathrm{i}\omega_k)\right)=0,\ k=\overline{0,n-1}. \quad (17)$$

Note that $G(\mathrm{i}\omega_k)$ is a constant complex number for any fixed ω_k. For fixed ω_k both numerator and denominator polynomials are linear combinations of the unknown process parameters. Thus, the set of equations (16), (17) represents a linear system of equations having 2n unknown coefficients. By solving this system of 2n linear equations, one obtains 2n coefficients of rational approximation (14).

It is convenient to represent this system of equations in the matrix form. This form is particularly suitable for the numerical evaluation of the unknown coefficients. Consider equations (16) and (17). Assuming the case $G(0)=0$, $a_0=1$ (other cases can be dealt in the similar fashion), for any $k=\overline{0,n-1}$ one obtains

$$G(\mathrm{i}\omega_k)\left(a_n(i\omega_k)^n+\ldots+a_1 i\omega_k+1\right)-b_{n-1}(i\omega_k)^{n-1}+\ldots+b_0=0,$$

or, introducing

$$R_{n,k}=\mathrm{Re}\{(i\omega_k)^n\},\quad I_{n,k}=\mathrm{Im}\{(i\omega_k)^n\},$$
$$GR_{n,k}=\mathrm{Re}\{G(i\omega_k)(i\omega_k)^n\},\quad GI_{n,k}=\mathrm{Im}\{G(i\omega_k)(i\omega_k)^n\},$$

one obtains

$$a_n GR_{n,k}+\ldots+a_1 GR_{1,k}-b_{n-1}R_{n-1,k}-\ldots-b_0 R_{0,k}=-GR_{0,k} \quad (18)$$

$$a_n GI_{n,k}+\ldots+a_1 GI_{1,k}-b_{n-1}I_{n-1,k}-\ldots-b_0 I_{0,k}=-GI_{0,k} \quad (19)$$

The obtained equations are conveniently rewritten in the following matrix form, which is easily solved using some of the modern computer algebra packages, in particular, introducing

$$\mathbf{M}=\begin{bmatrix} GR_{n,0} & \cdots & GR_{1,0} & -R_{n-1,0} & \cdots & -R_{0,0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ GR_{n,n-1} & \cdots & GR_{1,n-1} & -R_{n-1,n-1} & \cdots & -R_{0,n-1} \\ GI_{n,0} & \cdots & GI_{1,0} & -I_{n-1,0} & \cdots & -I_{0,0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ GI_{n,n-1} & \cdots & GI_{1,n-1} & -I_{n-1,n-1} & \cdots & -I_{0,0} \end{bmatrix},$$

$$\mathbf{b}=\begin{bmatrix} GR_{0,0} \\ \vdots \\ GR_{0,n-1} \\ GI_{0,0} \\ \vdots \\ GI_{0,n-1} \end{bmatrix},$$

one easily obtains the desired system of linear equations in matrix form

$$\mathbf{Mx}=\mathbf{b}, \quad (20)$$

where $\mathbf{x}$ is the vector of unknown parameters,

$$\mathbf{x}=\begin{bmatrix} a_{n-1} \\ \vdots \\ a_1 \\ b_{n-1} \\ \vdots \\ b_0 \end{bmatrix}.$$

It is important to mention that the selected set of points $\omega\in[\omega_0,\omega_2,\cdots,\omega_{n-1}]$ can produce a singular matrix of the set of equations (16), (17). In such a case, another, more appropriate set of points should be used. It is also significant to note that it is also possible to use more than n incident points in the selected set. The exact solution cannot be found in such a case. However, the best approximation, in the least-squares sense, can be found by means of pseudo-inversion.

III. Numerical Examples

Let us select several LFOS transfer functions and compare their Bode characteristics and responses to Heaviside excitation with those of the corresponding rational approximations determined on the basis of the set of linear equations (16) and (17). The exact characteristics are shown in blue line, while the approximations are plotted in red. For each of the examples considered below, a set of target frequency values used in (16) and (17) has been specified also.

Example 1. Consider fractional order system described by transfer function $G_1(s) = 1/(s^{3/2}+1)$, and consider its rational approximation obtained by interpolating frequency response in target points $\omega \in [0.01, 0.1, 0.5, 1, 5, 10, 100]$.

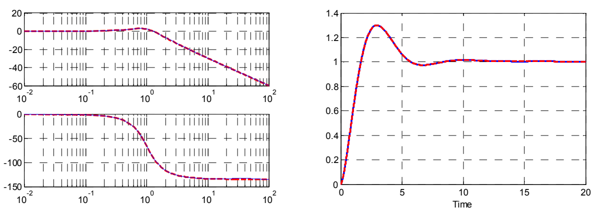

Fig 3. Frequency and time responses $G_1(s)$ (red) and $B_1(s)/A_1(s)$ (blue).

Example 2. Consider the fractional oscillator characterized by a strong resonant peak at unit angular frequency, $G_2(s) = 1/(s - \sqrt{2s} + 1)$, with $\omega \in [0.01, 0.1, 0.5, 1, 5, 10, 100]$.

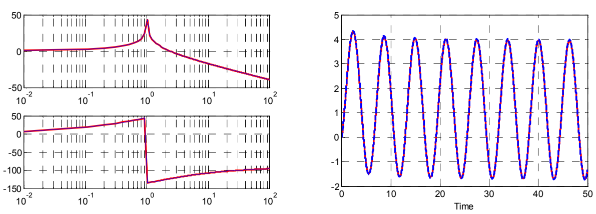

Fig 4. Frequency and time responses $G_2(s)$ (red) and $B_2(s)/A_2(s)$ (blue).

Example 3. Consider a process with $G_3(s) = \exp(-\sqrt{s})$.

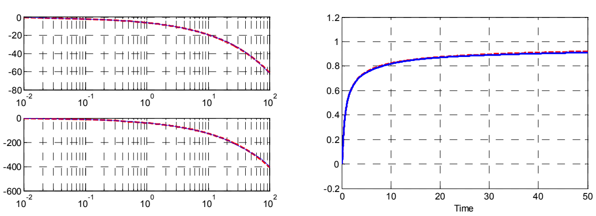

Fig 5. Frequency and time responses $G_3(s)$ (red) I $B_3(s)/A_3(s)$ (blue).

Processes with such a transfer function are common in analysis of distributed parameter systems, particularly those involving heat and mass transfer; see for example [22, 23]. Similar phenomena are, in a generalized form, also studied in [21].

Example 4. Fractional logarithmic filter with transfer function $G_4(s) = \ln(s)/s$ appears in the study of adaptive fractional systems, as can be seen from [24]. The incident frequencies in this particular case where $\omega \in [0.001, 0.01, 0.1, 0.5, 1, 5, 50]$.

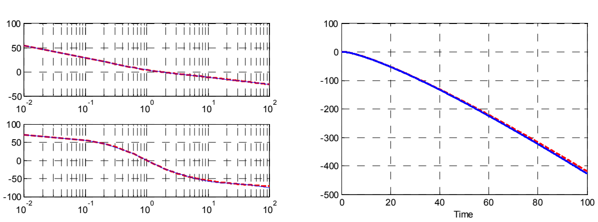

Fig 6. Frequency and time responses $G_3(s)$ (red) i $B_3(s)/A_3(s)$ (blue).

Example 5. The following example demonstrates a case when the process output is equal to the fractional semi-integral of a IIR filtered input. The actual IIR filter is, in fact, a differential compensator. $G_5(s) = \left(\frac{s+1}{0.1s+1}\right)^{0.5}$, $\omega \in [0.01, 0.1, 1, 3, 5, 10, 100]$.

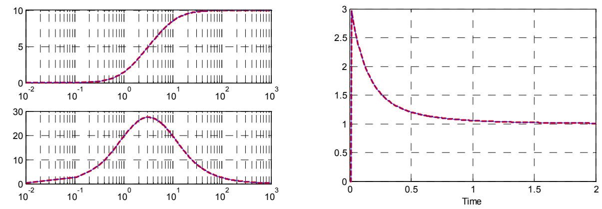

Fig 7. Frequency and time responses $G_5(s)$ (red) i $B_5(s)/A_5(s)$ (blue).

Example 6. $G_6(s) = (1 + 1/s + s^{1.2})/(0.1s+1)^{1.2}$, $\omega \in [0.5, 0.8, 1, 2, 5, 30, 100]$.

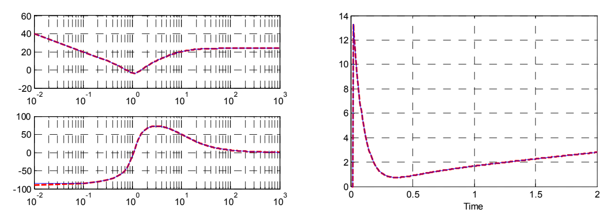

Fig 8. Frequency and time responses $G_6(s)$ (red) i $B_6(s)/A_6(s)$ (blue).

As the previous figures show, the rational approximations give adequate approximations for a wide range of LFOS. It should especially be mentioned that the corresponding frequency points are selected on the basis of the knowledge of Bode characteristics of LFOS transfer functions. In all previous examples the selected order n = 7 obviously can be lower, but under condition that the frequency characteristics in the selected frequency range are not violated. Also, for the selected set of frequencies in all these examples, the matrices of the sets of equations (16), (17) have been regular.

Obviously, since a rational transfer function is in question, applications of all techniques of discretization are possible, consequently adequate discrete models of LSF transfer functions within certain frequency range are available.

All of the time-domain responses presented above are obtained by means of direct integration in the complex domain. The interested reader is referred to [14].

IV. Conclusions

Owing to simplicity of application of the method of rational approximation of transfer functions of linear fractional order systems, this paper is dedicated to an analysis of the application of this approach for the purpose of discretization of linear fractional order systems. It should be emphasized that, since a rational transfer function of a continuous system is in question, application of all techniques of its discretization are possible, i.e. for discretization of linear fractional systems.

The most outstanding feature of the proposed method is its computational efficiency. The method is, in fact, very simple both conceptually and computationally. The obtained results are, as it can be seen from the previous examples quite satisfactory. The main drawback of the proposed method is that it is not possible to guarantee stability *a priori*, in other words no constraints on the coefficients are enforced. Indeed, the form of these constraints would be so complex, that their introduction would impair the established efficiency of the solution presented in the current paper.

A possible solution to this problem is to use the obtained coefficients as the initial guess for the more elaborate, non-convex optimization procedure. Such an approach would lose the desired computational efficiency, but would be able to give stronger stability guarantees.

REFERENCES

[1] I. Podlubny, Fractional Differential Equations, Academic Press, London, 1999.

[2] I. Podlubny, Fractional-Order Systems and $PI^{\lambda}D^{\mu}$ Controllers, IEEE Transaction on Automatic Control, vol. 44, 1999, pp. 208–214.

[3] R. Boxer, A Simplified Method of Solving Linear and Nonlinear Systems, Proceedings of the IRE, vol. 44, 1956, pp. 89-101.

[4] R. Boxer, A Note on Numerical Transform Calculus, Proceedings of the IRE, vol. 43, 1957, pp. 228-229.

[5] K. J. Åström, P. Hagander, J. Sterbnby, Zeros of sampled systems, IEEE Automatica vol. 20, 1984, pp. 31-38.

[6] K. J. Åström, B. Wittenmark, Computer Controlled Systems: Theory and Design, 3. ed., Prentice-Hall, 1997.

[7] M. R. Rapaić, T. B. Šekara, Novel direct optimal and indirect method for discretization of linear fractional systems, Electrical Engineering, vol 93, 2011, pp. 91-102.

[8] J. M. Smith, Mathematical Modeling and Digital Simulation for Engineers and Scientists, 2. ed., Wiley, New York, 1987.

[9] T. Šekara, New Transformation Polynomials for Discretization of Analogue Systems, Electrical Engineering, vol. 89, 2005, pp. 137-147.

T. B. Šekara, M. R. Stojić, Application of the α-approximation for discretization of analogue systems, Facta Universitatis, Ser: Elec. Energ vol. 3, pp. 571-586, 2005. http://factaee.elfak.ni.ac.yu/fu2k53/Sekara.pdf

[10] T. B. Šekara, L. S. Draganović, M. S. Stanković, Novel Approximations for Discretization of Continuous Systems, In: Proceedings of the XLVI ETRAN Conference, vol. 1, Bosnia and Herzegovina, 2002, pp 220-223 (in Serbian).

[11] T. B. Šekara, M. S. Stanković, Novel Transformation Polynomials for Discretization of Continuous Systems. In: Proc. of XLVII ETRAN Conference, vol. 1, 2003, pp. 247-250, Montenegro (in Serbian).

[12] T. B. Šekara, Indirect application of transformation polynomials for discretization offractional integrators (differentiators).In:Proc. of XLIL ETRAN Conference, vol.1, 2005, pp.234-237. Montenegro,(in. Serb.)

[13] T. B. Šekara, Fractional Transformations with Applications to Control Systems and Electrical Circuits. PhD Thesis, Faculty of Electrical Engineering, University of Belgrade (in Serbian), 2006.

[14] T. Dostál, J. Pospíšil, Switched circuits I, Rersearch Report No. FE-58, Department of Radioelectronics, TU Brno, 1985.

[15] G. Zhang, X. Chen, T. Chen, Digital Redesign via Generalized Bilinear Transform, International Journal of Control, vol. 82, 2009, pp. 741-754.

[16] K. Zaplatilek, M. Lares Efficient Algorithms of Direct and Inverse First-Order s-z Transformations. Radioengineering, vol. 10, 2001, pp. 6-10.

[17] Y. Choo, Computing Transformation Matrix for Bilinear s-z Transformation. IEICE Trans. Fundam., vol. E90-A, 2007, pp. 872-874.

[18] E. W. Bai,Z. Ding, Zeros of sampled data systems represented by FIR models,Automatica,vol. 36, 2000, pp. 121-123.

[19] J. Le Bihan, Novel Class of Digital Integrators and Differentiators, Electronic Letters, vol. 29, 1993, pp. 971–973.

[20] Milan R. Rapaić, Optimalno i suboptimalno upravljanje klasom sistema sa raspodeljenim parametrima, doktorska disertacija, Fakultet tehničkih nauka, Novi Sad, 2011

[21] Joel L. Schif, The Laplace transform – Theory and applications, Springer 1999

[22] Tomislav B. Šekara, Frakcioni sistemi upravljanja, Akademska misao, 2011

[23] Milan R. Rapaić, Alessandro Pisano, Adaptive on-line estimation of the commensurate order in fractional systems, The 5the IFAC, Nanjing, China, May 2012

[24] Tomislav B. Šekara, Milan R. Rapaić, Mihailo P. Lazarević, Indirect application of a method for discretization of linear fractional order systems, INDEL 2012.